Hall Crystal States at $\nu = 2$ and Moderate Landau Level Mixing

Ganpathy Murthy
Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506
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The $\nu = 2$ quantum Hall state at low Zeeman coupling is well-known to be a translationally invariant singlet if Landau level mixing is small. At zero Zeeman interaction, as Landau level mixing increases, the translationally invariant state becomes unstable to an inhomogeneous state. This is the first realistic example of a full Hall crystal, which shows the coexistence of quantum Hall order and density wave order. The full Hall crystal differs from the more familiar Wigner crystal by a topological property, which results in it having only linearly dispersing collective modes at small q, and no $q^{3/2}$ magnetophonon. I present calculations of the topological number and the collective modes.

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Integer quantum Hall states are some of the best understood, because single-particle approximations such as Hartree-Fock (HF) work very well for filled Landau levels (for an overview of the rich physics in these systems, see ref. [1]). The three important parameters that describe an integer state are the cyclotron frequency ω_c, the Zeeman coupling E_Z, and the dimensionless parameter characterizing the strength of the electron-electron interaction r_s. For $\nu = 2$ we have $r_s = \frac{e^2}{\varepsilon l_0 \hbar \omega_c}$. As r_s becomes large Landau level mixing increases.

We will work in the limit $E_Z = 0$, which is an excellent approximation for GaAs based systems, due to the reduction of the g factor, and the enhancement of the cyclotron frequency by band effects. In this limit, r_s survives as the only dimensionless parameter characterizing the system. The Hamiltonian in standard notation is

$$H = \sum n\hbar\omega_c a^\dagger_{\sigma,n,X} a_{\sigma,n,X} + \frac{1}{2L^2}\sum v(q) : \rho(\vec{q})\rho(-\vec{q}) : \tag{1}$$

where (σ, n, X) are spin, Landau level, and degeneracy indices for single-particle states in the Landau gauge, $a, a^\dagger$ are the fermion operators which destroy and create electrons in these single-particle states, and $\rho(\vec{q})$ is the density operator

$$\rho(\vec{q}) = \sum_{\sigma n_1 n_2 X} e^{-iq_x X} \rho_{n_1 n_2}(\vec{q}) a^\dagger_{\sigma,n_1,X-q_y l_0^2/2} a_{\sigma,n_2,X+q_y l_0^2/2} \tag{2}$$

Here l_0 is the magnetic length, and $\rho_{nn'}(\vec{q})$ is a matrix element.

What are the possible ground states of the system? At $\nu = 2$ the simplest possiblities are (i) Fill the $n = 0, \uparrow$ and $n = 0, \downarrow$ Landau levels to form the singlet state, or (ii) Fill $n = 0, 1$ for the $\uparrow$ spins only to form the fully polarized state. One can check that in the HF approximation, the fully polarized state becomes lower in energy than the singlet state for $r_s \geq 2.12$ for the Coulomb interaction, a transition first pointed out in a slightly different context by Giuliani and Quinn[2]. Finite thickness effects are modelled by an interaction $v(q) = \frac{e^2}{\varepsilon q} e^{-\lambda q}$ where the length λ is related to the sample thickness. This modifies the Coulomb interaction at large q and pushes the critical r_s for the singlet-fully polarized transition higher.

Translationally invariant states cannot take advantage of Landau level mixing (in HF), while inhomogeneous states can. Inhomogeneous states have been the subject of intense investigation in the early eighties in HF[3–7] in the context of their possible relevance to the fractional quantum Hall effect and the high-field Wigner crystal. In HF, such states are described by nonzero expectation values

$$\Delta_{\sigma n,\sigma' n'}(\vec{Q}) = \frac{2\pi l_0^2}{L^2}\sum e^{-iQ_x X} \langle a^\dagger_{\sigma n, X - Q_y l_0^2/2} a_{\sigma' n', X + Q_y l_0^2/2}\rangle \tag{3}$$

where L^2 is the area of the system, and $\vec{Q}$ are the set of reciprocal vectors of some lattice. Since we have two spin flavors, we can have states with spin mixing or not. Using these expectation values, one decouples the interaction term. One then performs a sequence of canonical transformations[5,6] which reduces the problem to diagonalizing a matrix for every point in the magnetic Brillouin zone. The dimension of this matrix is connected to the number of flux quanta per unit cell, and the number n_{LL} of LLs kept (typically I keep $n_{LL} = 10$ levels). If the flux per unit cell is $\phi = p\phi_0$ (here ϕ_0 is the flux quantum), then each Landau level breaks up into p nonoverlapping subbands. I have examined density wave states with two and three flux quanta per unit cell, with varying number of majority-spin subbands occupied, and for the square and triangular lattices. In the regime $3 \leq r_s \leq 9$ that I will mostly focus on, the triangular lattice with two flux quanta per unit cell, no spin-mixing, and a total polarization of half the maximal polarization turns out to have the lowest energy among all the density wave states that I studied. Anticipating the result that this is a Hall crystal, I will call this the partially polarized Hall crystal (PPHC). This state is the integer analog of the partially polarized density wave state for $\nu = 2/5$ that I have proposed[8] to explain the direct spin polarization

measurements of Kukushkin *et al*[9].

Figure 1 presents the results for the ground state energy for the translationally invariant singlet (S) state, the fully polarized (FP) state, and for the triangular PPHC state for the pure Coulomb interaction ($\lambda = 0$) and for a sample with finite thickness ($\lambda = 0.4l_0$). For $\lambda = 0$ the singlet is the lowest state among these for $r_s \leq 2.12$, while the PPHC state becomes the lowest energy state for $r_s \geq 6$. For $\lambda = 0.4l_0$ there is a direct transition from the S state to the PPHC state at $r_s \approx 4.5$, and the FP state is never the ground state.

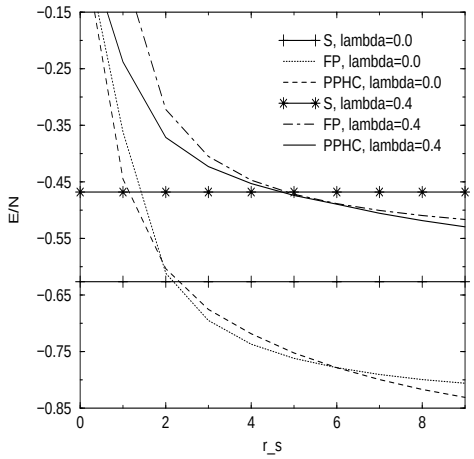

FIG. 1. Ground state energy per particle as a function of r_s for the singlet, fully polarized, and PPHC states for $\lambda = 0.0$ and $\lambda = 0.4$. The solid line is the singlet energy for $\lambda = 0.0$, while the dotted line is the singlet energy for $\lambda = 0.4$.

The PPHC state is distinguished from the S and the FP states by a density wave and a partial polarization. Another very important topological property of a crystal state was elucidated by Thouless and co-workers[10]. Suppose one considers a set of noninteracting electrons in the presence of a periodic potential. Then one can ask how much charge is transported when the lattice potential is adiabatically dragged by a lattice translation vector. It was shown[10] that in the thermodynamic limit, if the chemical potential lies in a gap, this charge transported is quantized, and characterized by an integer Chern index. In integer quantum Hall systems with a periodic potential, two integers, the quantized Hall conductance, and the above-mentioned Chern number, characterize each state.

More generally, one can ask whether crystalline and quantized Hall order can coexist, as was suggested by the cooperative ring-exchange theory[11]. This question was answered in the affirmative and the physical significance of the second integer was clarified by Tesanovic, Axel, and Halperin[12]. Given the two integers, the average density obeys the equation

$$\bar{\rho} = n_H \rho_\phi + n_C A_0^{-1} \tag{4}$$

where $n_H = h\sigma_{yx}/e^2$ is the integer characterizing the Hall conductance, ρ_ϕ is the density of flux quanta, A_0 is the area of the unit cell, and n_C is the Chern number describing the adiabatic transport of charge. The usual quantum Hall states have $n_H \neq 0$, but $n_C = 0$, while the usual Wigner crystals in the quantum Hall regime have $n_H = 0$, but $n_C \neq 0$.

Tesanovic *et al*[12] considered the case when there was a density wave, but with $n_C = 0$, which they called a *full Hall crystal*. They also labelled states with nonzero values of both integers as *partial Hall crystals* (not to be confused with partial polarization!). They explicitly constructed a (rotationally anisotropic) interaction with two-body and four-body parts for which they were able to show that the ground state was a full Hall crystal. One of their most important results concerns the low-energy collective modes of the various states[12]. It has long been known that the Wigner crystal has a single gapless magnetophonon collective mode with a dispersion of $\omega \propto q^{3/2}$ (for the long-range Coulomb interaction). This arises because the magnetic field mixes the usual ($B = 0$) linearly dispersing longitudinal and transverse lattice modes, both of which transport charge. After mixing, one mode is pushed up to ω_c, and the other is the magnetophonon. Tesanovic *et al* explicitly computed the collective modes for the full Hall crystal and showed that there are only *two linearly dispersing gapless modes*. Simply put, for $n_C = 0$, small q oscillations of the lattice produce no charge motion, and thus no magnetophonon.

In view of the above, it is interesting to ask for the values of the two integers characterizing the PPHC states. Here there are two spin flavors, and for charge motion we can treat the two additively. In this calculation (and in the collective mode calculation that follows), I employ a trick to keep only the active levels, the $n = 1$ majority-spin LL and the $n = 0$ minority-spin LL. The trick is to make $w_c \approx E_Z$ large compared to $e^2/\varepsilon l_0$, which eliminates LL-mixing. The PPHC state in this regime is adiabatically connected to the PPHC state at large r_s and $E_Z = 0$ (no gaps close as E_Z is decreased and LL-mixing is introduced). Therefore the two Chern numbers, and by implication, the structure of the low-energy collective charge modes, cannot change as one turns on LL-mixing. The integers can be easily computed by adapting the results of Tesanovic *et al* for the connection between n_H and n_C[12],

$$n_C = \frac{\phi}{p\phi_0} n_b - \frac{\phi}{\phi_0} n_H \tag{5}$$

where n_b is the number of filled nonoverlapping subbands (=4 including both spins for our case), and by borrowing the results of Yoshioka[13] (see also MacDonald[14]) for n_H for the triangular lattice periodic potential. To summarize Yoshioka's results, if, in a partially filled LL the electrons form an electron-like Wigner crystal, the contribution to n_H from this LL is zero, while if they form a hole-like Wigner crytal, they contribute 1 towards n_H. Which type of crystal the electrons like to form depends in turn on the sign of the effective potential $V(\vec{Q})$. In our case, it turns out that the $n = 0$ minority spin electrons form an electron-like Wigner crystal, while the $n = 1$ majority spin electrons form a hole-like Wigner crystal (the

$n = 0$ majority-spin electrons occupy a full LL, and therefore contribute $n_H = 1$). This leads to $n_H = 2$, $n_C = 0$, implying that this is a full Hall crystal. Note that the entire Hall current is carried by majority spin electrons, which has implications for spin-polarized transport.

Partially polarized square lattice crystalline states also exist. They are never lower in energy than the triangular ones for the model interaction I have chosen (though they are quite close). One can compute the Chern indices for this state as well, by adapting the results of Hatsugai and Kohmoto[15]. The difference here is that a fully gapped half-filled LL with two flux quanta per unit cell contributes ± 1 to n_H, depending on the sign of the effective potential. I find that this state is a *partial* Hall crystal, which has $n_H = 1$ and $n_C = 2$. This has the amusing feature that its Hall conductance is e^2/h despite a filling of $\nu = 2$!

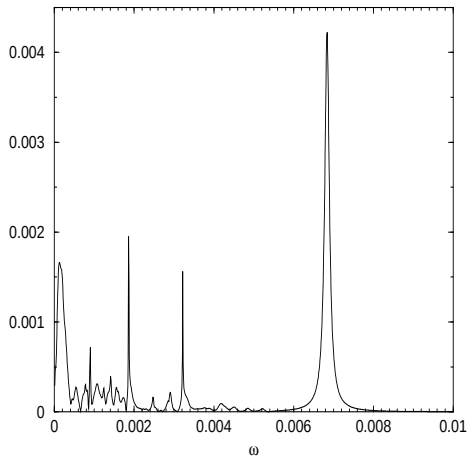

FIG. 2. Imaginary part of the charge response function as a function of ω for $q = 0.03$ in the triangular lattice PPHC. One optical mode carries most of the spectral weight, and three other sharp modes are visible at lower energies.

Let us now turn to collective excitations. In a Wigner crystal one finds a gapless $q^{3/2}$ magnetophonon, and in a full Hall crystal one finds two linearly dispersing gapless modes. Thus it is natural to expect both sets of modes in a partial Hall crystal. Furthermore, since we have an additional spin degree of freedom there should be more modes than in the spin-polarized case. I have computed the collective modes around the HF solution in the time-dependent HF approximation (TDHF) for both the triangular and square lattice crystalline states.

As explained by Cote and MacDonald[16], one can reduce the computation of collective modes to the diagonalization of a large matrix, from whose eigenvalues and eigenvectors one computes a response function. The poles of this response function give the physical collective modes[16]. The imaginary part of the charge response function as a function of ω for the triangular lattice is shown in Figure 2 at $ql_0 = 0.03$. The feature at $\omega \approx 0.007$ is an optical mode, while the sharp features at $\omega \approx 0.001,\ 0.002,\ 0.0033$ are gapless linearly dispersing charge modes. It can be seen that the optical mode has most of the weight. Now one follows these features as a function of q.

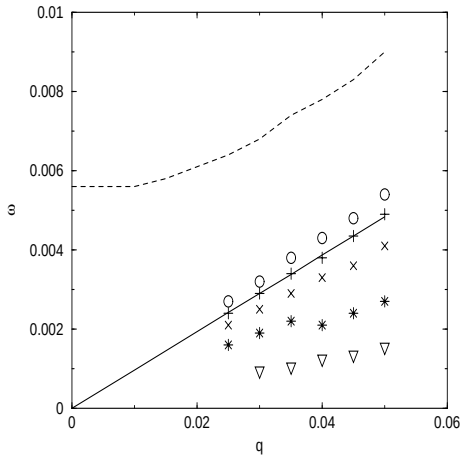

FIG. 3. The dispersion of some modes of the triangular lattice PPHC. All the linearly dispersing modes extrapolate to $\omega = 0$ at $q = 0$, showing that they are gapless (see, e.g., the linear fit to the + symbols). Since the spectral weight in the gapless modes decreases as some power of q for small q, their identification above the noise becomes problematic for $q \leq 0.02$.

The resulting set of dispersions for the triangular lattice is shown in Figure 3. As can be seen, there is no magnetophonon mode dispersing as $q^{3/2}$, while there are several linearly dispersing collective modes. All the linearly dispersing modes extrapolate to $\omega = 0$ at $q = 0$ within error, showing that they are indeed gapless. In Figure 4 we see the corresponding set of dispersions for the square lattice crystalline state. Here, in addition to the linearly dispersing gapless modes, the $q^{3/2}$ magnetophonon (symbolized by stars) does make an appearance, as expected for a partial Hall crystal.

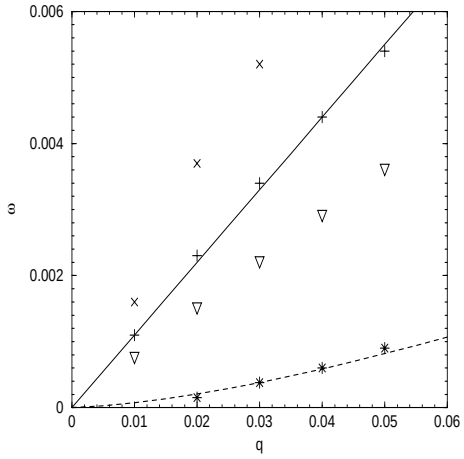

FIG. 4. The dispersion of some modes of the square lattice PPHC. All the linearly dispersing modes extrapolate to $\omega = 0$ at $q = 0$, showing that they are gapless (see the linear fit to the + symbols). The mode represented by the star symbol is the magnetophonon (see the $q^{3/2}$ dashed line fit).

Let us now turn to other related work. Very recently, Park and Jain[17] have performed a collective mode analysis (for zero thickness, $\lambda = 0$) of the S and FP states in the (r_s, E_Z) plane. Concentrating on their results for $E_Z = 0$, we see that both states are unstable for $r_s \approx 3$ and greater. To what state might this instability lead? The PPHC state is definitely not a candidate at this small r_s. I have found another state with equal occupations of the two spin flavors ($S_{z,total} = 0$), with a triangular lattice density wave with spin-mixing, whose ground state energy in HF is lower than that of the S and FP states for $1.75 \leq r_s \leq 2.7$. This state is a full Hall crystal with $n_H = 2$ and $n_C = 0$. Strangely enough, this state does not have the full symmetry of the triangular lattice, implying that the triangular lattice is not the optimal structure. Some variant of this spin-mixed density wave is likely to be the ground state for smaller r_s. These are likely to be spin-density waves but total singlets, raising the possibility of an inhomogeneous quantum Hall *antiferromagnet*[18] at $\nu = 2$ (an analogous state at the $\nu = 2$ edge has been explored recently[19]). The energy of this state is relatively higher than the PPHC state for larger r_s, so I believe that the PPHC state is still the ground state at large r_s. I am intensively exploring various spin-mixed states at small r_s to resolve this issue.

There are also experimental results on the $\nu = 2$ system. Recently, Eriksson *et al*[20] have measured the collective excitations of the $\nu = 2$ system by inelastic light scattering. They find that while they see a clear signature of the singlet nature of the ground state for $r_s \leq 3.3$, with a three-fold Zeeman split spin-density excitation, the situation changes for $r_s \geq 3.3$. Here they observe two nondispersing peaks which they interpret as two roton-like critical points in the dispersion around a singlet state which has been modified to include Fermi-liquid like parameters[20]. They further see the energies of these peaks decreasing linearly as r_s is increased, suggesting another transition. It is possible that the first transition is associated with a transition to the FP or a spin-mixed state, while the second could be the transition to the PPHC state. Certainly, as one approaches the transition to the PPHC one expects to see the would-be linearly dispersing modes soften if the transition is second-order or weakly first order (see Tesanovic *et al*[12] for an example of this). However, further measurements, specifically of the spin polarization, the Hall conductance, and the collective modes for $r_s \geq 6$ are needed to uniquely determine the nature of the state.

In summary, I have shown that there exist partially polarized Hall crystal states which are likely to be ground states of the $\nu = 2$ quantum Hall system at around $r_s \approx 6$. These are HF results, and subject to the usual caveat: Fluctuations beyond HF can alter the energies of various states. However, in fully gapped systems such as these, one expects HF to be not too far off. The triangular lattice PPHC state is, to my knowledge, the first realistic full Hall crystal, and has only linearly dispersing low energy modes for small q. The square lattice crystal state is a partial Hall crystal, with both a $q^{3/2}$ magnetophonon, and linearly dispersing modes. The square lattice PPHC state also has an unusual Hall conductance of e^2/h despite a filling of $\nu = 2$.

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